

XVIII. Observations on the Limits of Algebraical Equations; and a general Demonstration of Des Cartes's Rule for finding their Number of affirmative and negative Roots. By the Rev. Ifaac Milner, M. A. Fellow of Queen's College, Cambridge. Communicated by Anthony Shepherd, D. D. F. R. S. and Plumian Professor at Cambridge.

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§ 1. **T**HE investigations of the limits of equations is considered as one of the most important problems in algebra. The knowledge of them not only enables us to demonstrate many useful theorems in that science, but is also of material service in discovering the roots themselves. Mr. MACLAURIN has treated this subject very fully, both in his Algebra and in the Philosophical Transactions.

The substance of what he has delivered may be briefly expressed in the two following propositions.

1st. That any equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$ being proposed, if you take the fluxion of this equation,
and

and divide it by x , the resulting equation will have all its roots limits of the roots of the given equation.

2dly, If the terms of the proposed equation be multiplied into the terms of any arithmetical series, the resulting equation will also have its roots limits of the roots of the original equation.

§ 2. This second proposition, though admitted by all the eminent authors whom I have had an opportunity of consulting, certainly requires some restrictions. For example, the roots of the quadratic equation $x^2 - 2x - 3 = 0$ are 3, &c. - 1; multiply the terms of this equation into the terms of the arithmetical progression 1, 2, 3, respectively, and the resulting equation is $1 \times x^2 - 2 \times 2x - 3 \times 3 = 0$, the roots of which are $2 \pm \sqrt{13}$, neither of which are between the roots of the given quadratic.

Again, suppose the roots of the cubic equation $x^3 - px^2 + qx - r = 0$ to be $a, b, -c$, and it is possible that the equation $l + 3m \times x^3 - l + 2m \times px^2 + l + m \times qx - lr = 0$ may have no root between the quantities b and $-c$; and in general, if the roots of the equation (A) $x^n - px^{n-1} + qx^{n-2}$, &c. = 0 be supposed $a, b, c, -d, -e, -f$, &c. where a is the greatest root, b the next, and so on in order, the equation (B) $l + nm \times x^n - l + n - 1. mp \times x^{n-1} + l + n - 2. mq \times x^{n-2}$, &c. = 0 will not necessarily have any of its roots between the roots c and $-d$ of the original equation.

§ 3. It will not be difficult to see the reason of this, if we examine the demonstration, which is usually given us of this second proposition.

The roots of the biquadratic equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ are supposed to be a, b, c, d , and the results which arise by successively substituting them for x in $4x^3 - 3Ax^2 + 2Bx - C$ are supposed to be $-R, +S, -T, +Z$. From which MACLAURIN concludes, that when $abcd$ are substituted for x in the quantity $\overline{l + 4m \times x^4 - l \times 3m \times Ax^3 + l \times 2m \times Bx^2 - l \times m \times Cx + l \times D}$, the quantities that result will become $-mRx, +mSx, -mTx, +mZx$, where, says he, the signs being alternately negative and positive, it follows, that a, b, c, d , must be limits of the equation $\overline{l + 4m \times x^4 - l \times 3m \times Ax^3 + 8 \&c. = 0}$.

Here it is taken for granted, that the quantities $-mRx, +mSx, -mTx, +mZx$, are alternately negative and positive, which is not true, unless the roots a, b, c, d , be either all positive or all negative.

For suppose a, b, c , to be positive ^(a) quantities, and d a

(a) Philosophical Transactions, vol. XXXVI. Mr. MACLAURIN, who is here very diffuse upon this subject, never mentions any exception of this sort.

In his Algebra, art. 44. part 2. he says, he shall only treat of such equations as have their roots positive; but it may be observed, that his reasoning from art. 45. to 50. holds in all equations, the roots of which are real. The theorem in p. 182. of that treatise is not general, though applied in the eleventh chapter to the demonstration of NEWTON's rule for finding impossible roots in all equations.

negative

negative one; and then the four results will be $-mRa$, $+msb$, $-mTc$, $-mZd$.

§ 4. In general, the roots of the equation $\overline{nx^{n-1}-n-1} \cdot \overline{px^{n-2}+n-2} \cdot qx^{n-3}$, are always between the roots of the equation (A) because the roots of this last equation substituted successively for x in $\overline{nx^{n-1}-n-1} \cdot \overline{px^{n-2}+n-2}$ &c. always give the resulting quantities alternately negative and positive; but when the least of the affirmative roots, and the greatest of the negative roots of the equation (A) are substituted in (B) the quantities that result will necessarily have the same sign, and therefore it is possible, that no root of the equation (B) may lie between the least of the affirmative and the greatest of the negative roots of the equation (A).

§ 5. It is possible even, that the equation (B) may have imaginary roots, at the same time that all the roots of the equation A are real, which is contrary to what all algebraical writers have thought. For instance, the roots of the equation $x^2+6x-7=0$ are 7 and -1 , and if the terms of this equation be multiplied by 1, -1 , 3 (an arithmetical series where the common difference of the terms is equal to 2) the resulting equation will be $x^2-6x+21$, the roots of which are evidently impossible.

§ 6. However, the equation (B) can never have more than two imaginary roots, when the roots of the equation (A)

are real. For suppose these last roots to be $+a, +b, +c, +d, -e, -f, \&c.$ in their order from the greatest to the least, and since the results which arise from the successive substitution of these quantities are always alternately negative and positive, that case only excepted where d and $-e$ are substituted, it is manifest, that we shall always have $n-2$ of the roots of the equation (B) which will be limits of the equation (A).

§ 7. It is remarkable, that whenever the equation A has all its terms complete, its roots real, and some of them positive, and others negative, if $l+nm$ be assumed equal to 0, the equation B will always have one of its roots either greater than the greatest affirmative root, or less than the least negative root of the equation (A). Thus, in the quadratic $x^2+6x-7=0$, assume any arithmetical progression 0, 1, 2, the first term of which is equal to nothing, and the equation B in this case is $6x-14=0$ and $x=\frac{14}{6}$, which is greater than 1, the greatest affirmative root of the assumed equation.

§ 8. The roots of the equation (A) being still supposed $a, b, c, d, -e, -f, \&c.$ let m be taken equal to unity, and l any positive integer whatsoever, and in that case, two of the roots of the equation B will lie between the roots d and $-e$, and one of them will be positive, and the other negative.

For

For example, the quadratic equation $x^2 + 6x - 7 = 0$ has its roots 1 and -7 ; and if the terms of this equation be multiplied into 3, 2, 1; 4, 3, 2; or 5, 4, 3, successively, the resulting quadratic in every case will have its two roots between the roots of the given equation, and one of them will be positive, and the other negative.

§ 9. The equation B, which in the last article was deduced from the equation A by taking m equal to 1, and l any positive integer, may itself be treated in the same way, and the resulting equation will, *à fortiori*, have two of its roots between the roots d and $-e$ of the original equation, and one of them will be positive, and the other negative.

§ 10. Let $x^2 - px + q = 0$ represent any quadratic equation, the real roots of which are α and β ; suppose $x = \frac{1}{y}$, and we shall have $1 - py + qy^2 = 0$, the roots of which equation are $\frac{1}{\alpha}$, $\frac{1}{\beta}$. Let the root of the equation $2qy - p = 0$ be equal to $\frac{1}{A}$, and $\frac{1}{A}$ will always lie between the quantities $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and therefore one would think at first sight that the quantity A must always lie between α and β . But this would be contrary to what is proved in art. 7. In the present case A can never lie between α and β , unless these two quantities have the same sign, and it is obvious,

Mr. MILNER's Observations on the
obvious, that the same reasoning holds in equations of
higher dimensions.

These observations, as far as I know, are intirely new.
The fundamental proposition (§ 4.) was, in the year
1775, communicated to Dr. WARING, Lucasian profeffor
of mathematics in this university, and by him inserted
among the additions to his *Meditationes Algebraice*^(b).

§ 11. M. EULER, at the conclusion of his 13th chap.
Calcul. Different. has given a demonstration of DES
CARTES's rule for finding the number of affirmative and
of negative roots in any equation, the roots of which are
real. From what I have already said, his reasonings will
appear inconclusive, though I freely own, that what he
has done suggested the following different method.

Suppose (D) $L + mx + nx^2 + px^3 \dots + x^n = 0$, and the
roots of the equation (E) $m + 2nx \dots + nx^{n-1} = 0$ will
be limits of the roots of the equation (D); and therefore
there must be at least as many positive roots in the
equation (D) as there are in the equation (E). The same
may be said of the negative roots: for since every
root of the equation (E) lies between the different
roots of the equation (D), it is impossible that the num-
ber of roots should be less in either case. Suppose
L and Mx to be both positive, and since the last term in

(c) See the end of *Proprietates Curv.*

any equation is always the product of all the roots with their signs changed, the number of positive roots in each of the equations (D) and (E) must be even: therefore, the number of positive roots in (D) cannot exceed the number of those in (E) by unity; but there is in (D) one root more than in (E), and consequently it must be negative.

If both the terms L and mx are negative, because then the number of positive roots in (E) and (D) are even, it follows in the same way, that there is one negative root more in (D) than there is in (E).

And lastly, if the terms L and mx have different signs, for the same reasons there must be one positive root more in the equation (D) than there is in (E).

DES CARTES' rule is, that there are as many positive roots in any equation as there are changes in the signs of the terms from $+$ to $-$, or from $-$ to $+$, and that the remaining roots are negative. From what has been demonstrated it appears, that if this rule be true in the equation (E), it must hold also in the next equation (D) of superior dimensions; and as we know that it is true in simple and quadratic equations, it must therefore be true in cubics, in biquadratics, and so on.

This is one of the best rules we have in algebra. Dr. SAUNDERSON ^(c) saw such an infinity of cases in equa-

^(c) Vol. II. p. 683. Algebra.

tions of high dimenfions, that he fcarcely hoped for a general proof. MACLAURIN'S ^(d) method is plainly impracticable when the roots are numerous, and therefore this concife demonftration will perhaps be acceptable to mathematicians.

(d) Page 145. Algebra.

